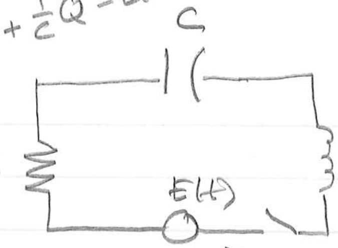
PAGE 1

ASIDE Q = charge on capacitor

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$



impressed Voltage

3.8: Forced Vibrations

Recall:

$$ma = F$$

$$mu'' = F_g + F_s + F_d + F(t)$$

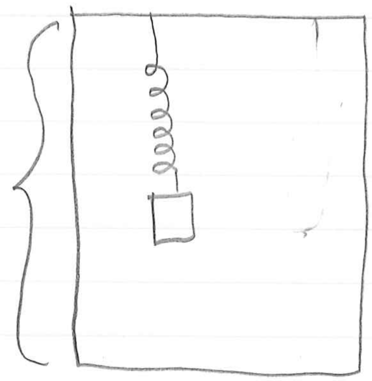
EXTERNAL FORCE

$$mu'' = mg - k(L+u) - \gamma u' + F(t)$$

$$\Rightarrow mu'' + \gamma u' + ku = \underbrace{mg - kL}_{0} + F(t)$$

$$\boxed{mu'' + \gamma u' + ku = F(t)}$$

External Force on the system

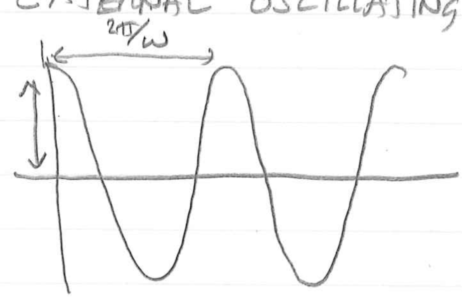


NOW WE WILL FOCUS ON EXTERNAL OSCILLATING FORCES:

$$F(t) = F_0 \cos(\omega t)$$

LARGEST MAGNITUDE OF FORCE

FREQUENCY OF FORCING



$\boxed{\text{CASE 1}}$: NO DAMPING, WITH FORCING

$$\gamma = 0$$

$$mu'' + ku = F_0 \cos(\omega t)$$

Solve $\boxed{\text{I}}$ $mr^2 + k = 0 \Rightarrow r^2 = -k/m \Rightarrow r_{1,2} = \pm\sqrt{\frac{k}{m}}\, i$

$\omega_0 = \sqrt{\frac{k}{m}}$ = NATURAL FREQUENCY OF SPRING.

$y_1 = \cos(\omega_0 t), \quad y_2 = \sin(\omega_0 t)$

$\boxed{\text{II}}$

- If $\omega \neq \omega_0$, then $Y_p(t) = A\cos(\omega t) + B\sin(\omega t)$

  AND SOLVING GIVES

  $A = \dfrac{F_0}{m(\omega_0^2 - \omega^2)}, \quad B = 0$

  $$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2-\omega^2)}\cos(\omega t)$$

RESONANCE

- If $\omega = \omega_0$, then $Y_p(t) = At\cos(\omega t) + Bt\sin(\omega t)$

  AND SOLVING GIVES

  $A = 0, \quad B = \dfrac{F_0}{2m\omega_0}$

  $$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}t\sin(\omega t)$$

SEE VISUALS...

Ex 1) $u'' + u = 60\cos(10t)$ ← ω

$\boxed{\text{I}}$ $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_1 = \cos(t),\ y_2 = \sin(t)$

$\boxed{\omega_0 = 1}$ NATURAL FREQUENCY

$\boxed{\text{II}}$ $A\cos(10t) + B\sin(10t) = Y(t)$

$-10A\sin(10t) + 10B\cos(10t) = Y'(t)$

$-100A\cos(10t) - 100B\sin(10t) = Y''(t)$

$Y'' + Y = 60\cos(10t)$

$-99A\cos(10t) - 99B\sin(10t) \stackrel{?}{=} 60\cos(10t)$

$A = -\frac{60}{99} = -\frac{20}{33},\quad B = 0$

$u(t) = c_1\cos(t) + c_2\sin(t) - \frac{20}{33}\cos(10t)$

$\omega_0 = 1 \qquad\qquad \omega = 10$

Ex 2) $u'' + u = 60\cos(t)$

$\boxed{\text{I}}$ $y_1 = \cos(t),\ y_2 = \sin(t),\ \omega_0 = 1$

$\boxed{\text{II}}$ $Y(t) = At\cos(t) + Bt\sin(t)$

MULT. BY t!!! BECAUSE $\cos(t)$ IS A HOMOGENEOUS SOL'N.

$Y'(t) = A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t)$

$Y''(t) = -A\sin(t) - A\sin(t) - At\cos(t) + B\cos(t) + B\cos(t) - Bt\sin(t)$

$Y'' + Y = 60\cos(t)$

$-2A\sin(t) + 2B\cos(t) - \cancel{At\cos(t)} - \cancel{Bt\sin(t)} + \cancel{At\cos(t)} + \cancel{Bt\sin(t)}$

$-2A\sin(t) + 2B\cos(t) \stackrel{?}{=} 60\cos(t)$

$2B = 60 \Rightarrow B = 30,\quad A = 0$

$u(t) = c_1\cos(t) + c_2\sin(t) + 30\sin(t)$

USEFUL IDENTITIES FOR GRAPHING

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{B-A}{2}\right)$$

$$\cos(A) - \cos(B) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{B-A}{2}\right)$$

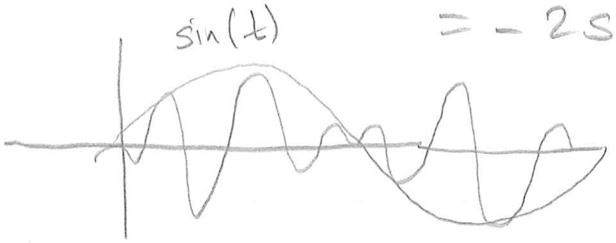
$$\sin(-x) = -\sin(x)$$

$$\cos(-x) = \cos(x)$$

$$\cos(7t) - \cos(5t) = 2\sin\left(\frac{12t}{2}\right)\sin\left(\frac{-2t}{2}\right)$$

$$= 2\sin(6t)\sin(-t)$$

$$= -2\sin(6t)\sin(t)$$



$$\cos(12t) + \cos(8t) = 2\cos(10t)\cos(2t)$$

$$\sin(24t) + \sin(18t) = 2\sin(21t)\cos(3t)$$

CASE 2 DAMPING & FORCING $(\gamma > 0)$

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

I HOMOGENEOUS

$$\begin{cases} \text{OVERDAMPED} & y_1 = e^{r_1 t}, \; y_2 = e^{r_2 t} \\ \text{CRITICALLY DAMPED} & y_1 = e^{rt}, \; y_2 = te^{rt} \\ \text{UNDERDAMPED} & y_1 = e^{\lambda t}\cos(\mu t), \; y_2 = e^{\lambda t}\sin(\mu t) \end{cases}$$

$$\underbrace{\lim_{t \to \infty} c_1 y_1(t) + c_2 y_2(t) = 0}$$

for ALL CASES, AND ALL INITIAL CONDITIONS, we say this is the "transient" part of the sol'n

II $Y_p = A\cos(\omega t) + B\sin(\mu t)$

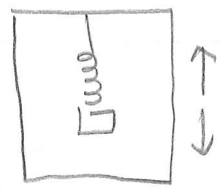
$$y(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{TRANSIENT PART OF SOL'N}} + \underbrace{A\cos(\omega t) + B\sin(\mu t)}_{\text{"STEADY STATE" OR "FORCED RESPONSE"}}$$

Ex] CONSIDER

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

with $m = 1\,\text{kg}$, $\gamma = 2\ \frac{N}{m/s}$, $k = 5\ N/m$, $F_0 = 10\,N$, $\omega = 1$

$$u'' + 2u' + 5u = 10\cos(t)$$



$10\cos(t)$

$\boxed{1}$ $r^2 + 2r + 5 = 0$

$$r = -\frac{2}{2} \pm \frac{1}{2}\sqrt{4-20} = -1 \pm \frac{1}{2}\sqrt{16}\, i = \overset{\lambda}{-1} \pm \underset{\mu = 2}{2i}$$

$$y_1 = e^{-t}\cos(2t), \quad y_2 = e^{-t}\sin(2t)$$

$\boxed{2}$ $y_p(t) = A\cos(t) + B\sin(t)$ $\quad\cdot 5 \leftarrow k$

$y_p' = -A\sin(t) + B\cos(t)$ $\quad\cdot 2 \leftarrow \gamma$

$y_p'' = -A\cos(t) - B\sin(t)$ $\quad\cdot 1 \leftarrow m$

$$(-A + 2B + 5A)\cos(t) + (-B - 2A + 5B)\sin(t)$$
$$= (2B + 4A)\cos(t) + (-2A + 4B)\sin(t) \overset{?}{=} 10\cos(t)$$

$2B + 4A = 10 \Rightarrow B = 5 - 2A$

$-2A + 4B = 0 \Rightarrow -A + 2B = 0$

COMBINE $\Rightarrow -A + 2(5 - 2A) \overset{?}{=} 0$

$-A + 10 - 4A = 0$

$10 = 5A \Rightarrow A = 2$

$\boxed{B = 1}$

$$\boxed{y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \underbrace{2\cos(t) + \sin(t)}_{\text{STEADY STATE SOL'N!}}}$$

SHOW PICTURES

$$\sqrt{5}\,\cos(t - \delta)$$